ISRAEL JOURNAL OF MATHEMATICS 143 (2004), 223–238

ℓ^1-SPREADING MODELS IN MIXED TSIRELSON SPACE

BY

DENNY H. LEUNG

Department of Mathematics, National University of Singapore
2 Science Drive 2, Singapore 117543
e-mail: matlhh@nus.edu.sg

AND

WEE-KEE TANG

Mathematics and Mathematics Education, National Institute of Education
Nanyang Technological University, 1 Nanyang Walk, Singapore 637616
e-mail: wktang@nie.edu.sg

ABSTRACT

Suppose that $(\mathcal{F}_n)_{n=1}^{\infty}$ is a sequence of regular families of finite subsets of $\mathbb{N}$ and $(\theta_n)_{n=1}^{\infty}$ is a nonincreasing null sequence in $(0,1)$. The mixed Tsirelson space $T[(\theta_n, \mathcal{F}_n)_{n=1}^{\infty}]$ is the completion of c_{00} with respect to the implicitly defined norm

$$\|x\| = \max\{\|x\|_{c_0}, \sup_n \sup \theta_n \sum_{i=1}^{k} \|E_i x\|\},$$

where the last supremum is taken over all sequences $(E_i)_{i=1}^{k}$ in $[\mathbb{N}]^{<\infty}$ such that $\max E_i < \min E_{i+1}$ and $\{\min E_i : 1 \le i \le k\} \in \mathcal{F}_n$. Necessary and sufficient conditions are obtained for the existence of higher order ℓ^1-spreading models in every subspace generated by a subsequence of the unit vector basis of $T[(\theta_n, \mathcal{F}_n)_{n=1}^{\infty}]$.

1. Preliminaries

Mixed Tsirelson spaces were first introduced by Argyros and Deliyanni [2]. They furnish a central class of examples in the recent development of the structure theory of Banach spaces. In [9], the authors computed the Bourgain ℓ^1-indices of mixed Tsirelson spaces. A stronger measure of the finite dimensional ℓ^1-structure of a Banach space is the presence of (higher order) ℓ^1-spreading

Received May 19, 2003

models. Kutzarova and Lin [7] showed that the Schlumprecht space [11], a fundamental example that opened the door to much of the recent progress in the structure theory of Banach spaces, contains an ℓ^1-spreading model. Subsequently, Argyros, Deliyanni and Manoussakis [4] showed that if $\theta_{n+m} \geq \theta_n\theta_m$ and $\lim_n \theta_n^{1/n} = 1$, then the mixed Tsirelson space $T[(\theta_n, \mathcal{S}_n)_{n=1}^{\infty}]$ contains ℓ^1-$\mathcal{S}_\omega$-spreading models hereditarily. In the present paper, we consider general mixed Tsirelson spaces $T[(\theta_n, \mathcal{F}_n)_{n=1}^{\infty}]$ and obtain necessary and sufficient conditions for the existence of higher order ℓ^1-spreading models in every subspace generated by a subsequence of the unit vector basis.

We set the notation in the remainder of the section. Endow the power set of $\mathbb{N}$, identified with $2^{\mathbb{N}}$, with the product topology. If M is an infinite subset of $\mathbb{N}$, denote the set of all finite, respectively infinite, subsets of M by $[M]^{<\infty}$, respectively $[M]$. A family $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ is said to be **hereditary** if $G \subseteq F \in \mathcal{F}$ implies $G \in \mathcal{F}$. It is **spreading** if whenever $F = \{n_1, \ldots, n_k\} \in \mathcal{F}$, $n_1 < \cdots < n_k$, and $m_1 < \cdots < m_k$ satisfy $m_i \geq n_i$, $1 \leq i \leq k$, then $\{m_1, \ldots, m_k\} \in \mathcal{F}$. A **regular** family is one that is hereditary, spreading and compact (as a subset of the topological space $[\mathbb{N}]^{<\infty}$). If E and F are finite subsets of $\mathbb{N}$, we write $E < F$, respectively $E \leq F$, to mean $\max E < \min F$, respectively $\max E \leq \min F$ ($\max \emptyset = 0$ and $\min \emptyset = \infty$). We abbreviate $\{n\} < E$ and $\{n\} \leq E$ to $n < E$ and $n \leq E$ respectively. Given $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$, a sequence of finite subsets $\{E_1, \ldots, E_n\}$ of $\mathbb{N}$ is said to be **$\mathcal{F}$-admissible** if $E_1 < \cdots < E_n$ and $\{\min E_1, \ldots, \min E_n\} \in \mathcal{F}$. If $\mathcal{M}$ and $\mathcal{N}$ are regular subsets of $[\mathbb{N}]^{<\infty}$, we let

$$\mathcal{M}[\mathcal{N}] = \{\bigcup_{i=1}^{k} F_i : F_i \in \mathcal{N} \text{ for all } i \text{ and } \{F_1, \ldots, F_k\} \text{ is } \mathcal{M}\text{-admissible}\}.$$

Given a sequence of regular families $(\mathcal{M}_i)$, we define inductively $[\mathcal{M}_1, \mathcal{M}_2] = \mathcal{M}_1[\mathcal{M}_2]$ and $[\mathcal{M}_1, \ldots, \mathcal{M}_{i+1}] = [\mathcal{M}_1, \ldots, \mathcal{M}_i][\mathcal{M}_{i+1}]$. Also, let

$$(\mathcal{M}_1, \ldots, \mathcal{M}_k) = \{\bigcup_{i=1}^{k} M_i : M_i \in \mathcal{M}_i, M_1 < \cdots < M_k\}.$$

We abbreviate the k-fold constructions $[\mathcal{M}, \ldots, \mathcal{M}]$ and $(\mathcal{M}, \ldots, \mathcal{M})$ as $[\mathcal{M}]^k$ and $(\mathcal{M})^k$ respectively. Of primary importance are the Schreier classes as defined in [1]. Let $\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$ and $\mathcal{S}_1 = \{F \subseteq \mathbb{N} : |F| \leq \min F\}$. Here $|F|$ denotes the cardinality of F. The higher Schreier classes are defined inductively as follows: $\mathcal{S}_{\alpha+1} = \mathcal{S}_1[\mathcal{S}_\alpha]$ for all $\alpha < \omega_1$. If α is a countable limit ordinal, choose a sequence (α_n) strictly increasing to α and set

$$\mathcal{S}_\alpha = \{F : F \in \mathcal{S}_{\alpha_n} \text{ for some } n \leq \min F\}.$$

It is clear that $\mathcal{S}_\alpha$ is a regular family for all $\alpha < \omega_1$. Given a nonzero countable ordinal α whose Cantor normal form is $\alpha = \omega^{\beta_1} \cdot m_1 + \cdots + \omega^{\beta_n} \cdot m_n$, we let $\mathcal{R}_\alpha$ be the regular family $((\mathcal{S}_{\beta_n})^{m_n}, \ldots, (\mathcal{S}_{\beta_1})^{m_1})$. If $\mathcal{F}$ is a closed subset of $[\mathbb{N}]^{<\infty}$, let $\mathcal{F}'$ be the set of all limit points of $\mathcal{F}$. Define a transfinite sequence of sets $(\mathcal{F}^{(\alpha)})_{\alpha<\omega_1}$ as follows: $\mathcal{F}^{(0)} = \mathcal{F}$, $\mathcal{F}^{(\alpha+1)} = (\mathcal{F}^{(\alpha)})'$ for all $\alpha < \omega_1$; $\mathcal{F}^{(\alpha)} = \bigcap_{\beta<\alpha} \mathcal{F}^{(\beta)}$ if α is a countable limit ordinal. If $\mathcal{F}$ is regular, we let $\iota(\mathcal{F})$ be the unique ordinal α such that $\mathcal{F}^{(\alpha)} = \{\emptyset\}$. It is well known that $\iota(\mathcal{S}_\gamma) = \omega^\gamma$ for all $\gamma < \omega_1$ [1, Proposition 4.10]. Also, $\iota((\mathcal{M}, \mathcal{N})) = \iota(\mathcal{N}) + \iota(\mathcal{M})$ and $\iota(\mathcal{M}[\mathcal{N}]) \leq \iota(\mathcal{N}) \cdot \iota(\mathcal{M})$ [8, Proposition 10]. In particular, $\iota(\mathcal{R}_\alpha) = \alpha$.

If $\mathcal{F}$ is a regular family and K is a positive constant, we say that a normalized sequence (x_n) in a Banach space is an **ℓ^1-$\mathcal{F}$-spreading model with constant** K if $\|\sum_F a_n x_n\| \geq K^{-1} \sum_F |a_n|$ for all $F \in \mathcal{F}$ and all sequences of scalars (a_n). We refer to [6] for the definitions and in depth discussions of the ℓ^1-indices $I(X)$, $I(X, K)$, $I_b(X)$ and $I_b(X, K)$ of a Banach space X (assumed to have a basis in the last two). Suffice it to say that if X contains an ℓ^1-$\mathcal{F}$-spreading model with constant K, then $I(X, K) \geq \iota(\mathcal{F})$. Moreover, if the spreading model is a block basis of the basis of X, then $I_b(X, K) \geq \iota(\mathcal{F})$.

Let c_{00} be the vector space of all finitely supported real sequences and let (e_k) be the standard unit vector basis of c_{00}. For $E \in [\mathbb{N}]^{<\infty}$ and $x = \sum a_k e_k \in c_{00}$, let $Ex = \sum_{k \in E} a_k e_k$. Given a sequence of regular families $(\mathcal{F}_n)_{n=1}^\infty$ and a nonincreasing null sequence $(\theta_n)_{n=1}^\infty$ in $(0, 1)$, the **mixed Tsirelson space** $T[(\theta_n, \mathcal{F}_n)_{n=1}^\infty]$ is the completion of c_{00} under the implicitly defined norm

$$\|x\| = \max\{\|x\|_{c_0}, \sup_n \sup \theta_n \sum_{i=1}^k \|E_i x\|\}, \tag{1}$$

where the last supremum is taken over all $\mathcal{F}_n$-admissible sequences $(E_i)_{i=1}^k$.

Throughout the paper, we consider a fixed mixed Tsirelson space $X = T[(\theta_n, \mathcal{F}_n)_{n=1}^\infty]$. Set $\alpha_n = \iota(\mathcal{F}_n)$ for all n and let $\alpha = \sup_n \alpha_n$. To avoid trivial cases, we will assume that $\alpha_n > 1$ for all n. The following fundamental set theoretic dichotomy due to Gasparis will be used repeatedly.

THEOREM 1 ([5, Theorem 1.1]): *Let $\mathcal{F}$ and $\mathcal{G}$ be hereditary families of finite subsets of $\mathbb{N}$ and N an infinite subset of $\mathbb{N}$. Then there exists $M \in [N]$ such that either $\mathcal{G} \cap [M]^{<\infty} \subseteq \mathcal{F}$ or $\mathcal{F} \cap [M]^{<\infty} \subseteq \mathcal{G}$.*

Note that if $\mathcal{G}$ is a regular family, then $\iota(\mathcal{G} \cap [M]^{<\infty}) = \iota(\mathcal{G})$ for all $M \in [\mathbb{N}]$. Thus if $\mathcal{F}$ and $\mathcal{G}$ are regular families such that $\iota(\mathcal{F}) < \iota(G)$, then for any $N \in [\mathbb{N}]$, there exists $M \in [N]$ such that $\mathcal{F} \cap [M]^{<\infty} \subseteq \mathcal{G}$.

PROPOSITION 2: *If $\alpha = \alpha_n$ for some n or if α is not of the form ω^{ω^ξ}, $\xi < \omega_1$, then X contains ℓ^1-$\mathcal{R}_{\alpha^k}$-spreading models hereditarily for all $k \in \mathbb{N}$. However, it does not contain any ℓ^1-$\mathcal{R}_{\alpha^\omega}$-spreading model.*

Proof: Let (x_m) be a normalized block sequence in X. Under the hypothesis, for any $k \in \mathbb{N}$, there exist $n, i \in \mathbb{N}$ such that $\alpha^k < \alpha_n^i$. Then $\iota(\mathcal{R}_{\alpha^k}) < \iota([\mathcal{F}_n]^i)$. By Theorem 1 and the subsequent remark, there exists $M \in [\mathbb{N}]^{<\infty}$ such that $\mathcal{R}_{\alpha^k} \cap [M]^{<\infty} \subseteq [\mathcal{F}_n]^i$. We claim that $(x_m)_{m\in M}$ is an ℓ^1-$\mathcal{R}_{\alpha^k}$-spreading model with constant $1/\theta^i$. Indeed, suppose that $M = (m_j)$ and $F \in \mathcal{R}_{\alpha^k}$; then $\{m_j : j \in F\} \in \mathcal{R}_{\alpha^k} \cap [M]^{<\infty} \subseteq [\mathcal{F}_n]^i$. As a result, $\{\operatorname{supp} x_{m_j} : j \in F\}$ is $[\mathcal{F}_n]^i$-admissible. Therefore, for all $(a_j) \in c_{00}$,

$$\left\| \sum_{j\in F} a_j x_{m_j} \right\| \geq \theta_n^i \sum_{j\in F} \|a_j x_{m_j}\| = \theta_n^i \sum_{j\in F} |a_j|.$$

On the other hand, $I_b(X) = \alpha^\omega$ [9, Theorem 15]. If $\alpha \geq \omega$, then $I(X) = I_b(X) = \alpha^\omega$ by [6, Corollary 5.13]. By [6, Lemma 5.11], $I(X, K) < \alpha^\omega$ for all $K \geq 1$. It follows that X does not contain an ℓ^1-$\mathcal{R}_{\alpha^\omega}$-spreading model. If $\alpha < \omega$, then $\alpha^\omega = \omega$ since we are assuming that $\alpha > 1$. If (x_n) is an ℓ^1-$\mathcal{S}_1$-spreading model in X, then there is a subsequence (x_{n_k}) such that $(x_{n_{2k}} - x_{n_{2k+1}})$ is equivalent to a block basis of the unit vector basis (e_k) of X. It is easily checked that $(x_{n_{2k}} - x_{n_{2k+1}})$ is an ℓ^1-$\mathcal{S}_1$-spreading model. Thus $\omega \leq I_b(X, K)$ and hence $I_b(X) = I_b(X, K)$, contrary to [6, Lemma 5.7]. ∎

2. Higher order ℓ^1-spreading models

Henceforth, we assume that $\alpha \neq \alpha_n$ for any n and $\alpha = \omega^{\omega^\xi}$ for some $0 < \xi < \omega_1$. For a nonzero ordinal α with Cantor normal form $\omega^{\beta_1} \cdot m_1 + \cdots + \omega^{\beta_n} \cdot m_n$, let $\ell(\alpha) = \beta_1$. Given $m \in \mathbb{N}$ and $\varepsilon > 0$, define

$$\gamma = \gamma(\varepsilon, m) = \max\{\ell(\alpha_{n_s} \dots \alpha_{n_1}) : \varepsilon\theta_{n_1} \cdots \theta_{n_s} > \theta_m\} \quad (\max\emptyset = 0).$$

We say that the space X satisfies (†) if

> there exists $\varepsilon > 0$ such that for all $\beta < \omega^\xi$, there exists $m \in \mathbb{N}$ satisfying $\gamma(\varepsilon, m) + 2 + \beta < \ell(\alpha_m)$.

It was proved in [9] that condition (†) is sufficient for X to have a large ℓ^1-index.

THEOREM 3 ([9, Theorem 17]): *Assume that $\xi \neq 0$. If X satisfies (†), then $I(X) = \omega^{\omega^{\xi\cdot 2}}$.*

Remark: It was shown in [9, Corollary 18] that (†) holds if ξ is a limit ordinal.

Observe that if X contains an ℓ^1-$\mathcal{S}_{\omega^\xi}$-spreading model, then it actually contains ℓ^1-$\mathcal{F}_n[\mathcal{S}_{\omega^\xi}]$-spreading models for all n. In this case, it follows that $I(X) = \omega^{\omega^\xi \cdot 2}$. Hence the next result strengthens Theorem 3.

THEOREM 4: *Suppose that $0 < \xi < \omega_1$ and (†) holds. Then for any subsequence $(e_n)_{n\in M}$ of the unit vector basis (e_n) of X, $[(e_n)_{n\in M}]$ contains an ℓ^1-$\mathcal{S}_{\omega^\xi}$-spreading model.*

The construction, using interlaced layers of vectors of differing complexities, is based on the method pioneered by Kutzarova and Lin ([7]) and subsequently refined and extended by Argyros et al. ([3]). As in [9], we calculate the norms of vectors in X by means of admissible trees. Let us recall the relevant procedure and set the notation. A **tree** in $[\mathbb{N}]^{<\infty}$ is a finite collection of elements (E_i^m), $0 \le m \le r$, $1 \le i \le k(m)$, in $[\mathbb{N}]^{<\infty}$ so that for each m, $E_1^m < E_2^m < \cdots < E_{k(m)}^m$, and that every E_i^{m+1} is a subset of some E_j^m. The elements E_i^m are called **nodes** of the tree. Any node E_i^m is said to be of **level** m. Nodes at level 0 are called **roots**. If $E_i^n \subseteq E_j^m$ and $n > m$, we say that E_i^n is a **descendant** of E_j^m and E_j^m is an **ancestor** of E_i^n. If, in the above notation, $n = m+1$, then E_i^n is said to be an **immediate successor** of E_j^m, and E_j^m the **immediate predecessor** of E_i^n. Nodes with no descendants are called **terminal nodes** or **leaves** of the tree. The set of all leaves of a tree $\mathcal{T}$ is denoted by $\mathcal{L}(\mathcal{T})$. A tree (E_i^m), $0 \le m < r$, $1 \le i \le k(m)$, is $(\mathcal{F}_n)$-admissible if $k(0) = 1$ and for every m and i, the collection (E_j^{m+1}) of all immediate successors of E_i^m is an $\mathcal{F}_n$-admissible collection for some $n \in \mathbb{N}$. Given an $(\mathcal{F}_n)$-admissible tree (E_i^m), we define the **history** of the individual nodes inductively as follows. Let $h(E_1^0) = (0)$. If $h(E_i^m)$ has been defined and the collection (E_j^{m+1}) of all immediate successors of E_i^m forms an $\mathcal{F}_n$-admissible collection, then define $h(E_j^{m+1})$ to be the $(m+2)$-tuple $(h(E_i^m), n)$. Finally, assign ((θ_n)-compatible) **tags** to the nodes by defining $t(E_i^m) = \prod_{j=0}^m \theta_{n_j}$ if $h(E_i^m) = (n_0, n_1, \ldots, n_m)$ $(\theta_0 = 1)$. If $x \in c_{00}$ and $\mathcal{T}$ is an $(\mathcal{F}_n)$-admissible tree, let $\mathcal{T}x = \sum t(E)\|Ex\|_{c_0}$, where the sum is taken over all leaves in $\mathcal{T}$. It is easily observed that $\|x\| = \max\{\mathcal{T}x : \mathcal{T} \text{ is an } (\mathcal{F}_n)\text{-admissible tree}\}$.

We are now ready to set up for the main step of the calculation. Let $\varepsilon \in (0,1)$ be given. For $r \in \mathbb{N}$, let $\mathcal{N}_r = \{(0, n_1, \ldots, n_s) : \varepsilon\theta_{n_1}\cdots\theta_{n_s} > \theta_r\}$. Then $\gamma(\varepsilon, r) = \max\{\ell(\alpha_{n_s}\ldots\alpha_{n_1}) : (0, n_1, \ldots, n_s) \in \mathcal{N}_r\}$. Assume $\delta \in (0,1)$, $p < q$

and η are given such that $\gamma(\varepsilon,p) < \eta < \omega^\xi$. Let

$$K_{\delta,p,\eta} = \{(0,n_1,\ldots,n_s) : \theta_{n_1}\cdots\theta_{n_s} > \delta\theta_p,\ \ell(\alpha_{n_s}\ldots\alpha_{n_1}) < \eta\}.$$

Also assume that $M \in [\mathbb{N}]$ satisfies $[\mathcal{F}_{n_1},\ldots,\mathcal{F}_{n_s}] \cap [M]^{<\infty} \subseteq \mathcal{S}_\eta$ whenever $(0,n_1,\ldots,n_s) \in K_{\delta,p,\eta}$. Suppose that vectors x_1 and x_2 are given so that

$$x_1 = \theta_p^{-1}\sum_{i=1}^r a_i e_{m_i}, \quad x_2 = \sum_{i=1}^r a_i z_i, \quad x = x_1 + x_2,$$

and

$$\begin{aligned} \|x_1\|_{\mathcal{S}_\eta} &\le \frac{\delta}{|K_{\delta,p,\eta}|+1},\\ \{m_1,m_2,\ldots,m_r\} &\in \mathcal{S}_{\eta+1}\cap[M]^{<\infty},\\ \|x_1\|_{\ell^1} &= 1/\theta_p,\\ m_1 < z_1 < &\cdots < m_r < z_r. \end{aligned}$$

If $y = \sum a_k e_k \in c_{00}$ and $\mathcal{F}$ is a regular family, let $\|y\|_{\mathcal{F}} = \sup_{F\in\mathcal{F}}\sum_{k\in F}|a_k|$.

PROPOSITION 5: *Let x be given as above. For any admissible tree $\mathcal{T}$, there exist an admissible tree $\mathcal{T}'$ and disjoint sets J_1 and J_2 such that*

(1) *$\mathcal{T}'$ is (p,q)-restricted, i.e., for all $E \in \mathcal{L}(\mathcal{T}')$, there exists $G \in \mathcal{T}'$ containing E such that $h(G) \in \mathcal{N}_q \setminus \mathcal{N}_p$,*

(2)
$$\mathcal{T}x \le \mathcal{T}\bigg(\sum_{i\in J_1}\frac{a_i}{\theta_p}e_{m_i} + \sum_{i\in J_2}a_i z_i\bigg) + \mathcal{T}'x_2 + \delta + \frac{\theta_q}{\varepsilon\theta_p}.$$

Proof: Choose $m_{r+1} > \max\operatorname{supp} z_r$. We may assume without loss of generality that the root of $\mathcal{T}$ is the integer interval $[m_1, m_{r+1}]$, that every node in $\mathcal{T}$ is an integer interval, and that every leaf in $\mathcal{T}$ is a singleton. For each $i \le r$, let $\mathcal{E}_i = \{E \in \mathcal{L}(\mathcal{T}) : E \subseteq \operatorname{supp} z_i\}$. Define

$$\begin{aligned} I_1 &= \{i : \mathcal{E}_i \ne \emptyset,\ \{m_i\} \in \mathcal{L}(\mathcal{T})\},\\ I_2 &= \{i : \mathcal{E}_i \ne \emptyset,\ \{m_i\} \notin \mathcal{L}(\mathcal{T})\}, \quad \text{and}\\ I_3 &= \{i : \mathcal{E}_i = \emptyset,\ \{m_i\} \in \mathcal{L}(\mathcal{T})\}. \end{aligned}$$

If $\{m_i\} \in \mathcal{L}(\mathcal{T})$, we write t_i for the tag $t(\{m_i\})$. Observe that

$$(2)\qquad \begin{aligned} \mathcal{T}x &= \sum_{E\in\mathcal{L}(\mathcal{T})} t(E)\|Ex\|_{c_0}\\ &\le \sum_{i\in I_1\cup I_3} t_i\frac{|a_i|}{\theta_p} + \sum_{i\in I_1\cup I_2}\sum_{E\in\mathcal{E}_i}|a_i|t(E)\|Ez_i\|_{c_0}. \end{aligned}$$

For each $i \in I_1$, let F_i be the smallest (by set inclusion) node in $\mathcal{T}$ such that $\{m_i, m_{i+1}\} \subseteq F_i$, then let G_i be the immediate successor of F_i containing m_i. Note that if $i_1, i_2 \in I_1$ and $i_1 < i_2$, then $G_{i_1} \neq G_{i_2}$. For otherwise, since $G_{i_1} = G_{i_2}$ is an integer interval, $\{m_{i_1}, m_{i_1+1}\} \subseteq G_{i_1} \subsetneqq F_{i_1}$, contrary to the choice of F_{i_1}. Subdivide I_1 into I_1', I_1'', and I_1''' according to whether $h(G_i) \in \mathcal{N}_p$, $h(G_i) \in \mathcal{N}_q \backslash \mathcal{N}_p$, or $h(G_i) \notin \mathcal{N}_q$. Suppose $i \in I_1'$. Then $h(G_i) = (0, n_1, \ldots, n_s) \in \mathcal{N}_p$. It follows that $\theta_{n_1} \cdots \theta_{n_s} > \delta\theta_p$ and $\ell(\alpha_{n_s} \cdots \alpha_{n_1}) \leq \gamma(\varepsilon, p) < \eta$. Thus $h(G_i) \in K_{\delta,p,\eta}$. Hence

$$(3) \qquad \begin{aligned} \sum_{i \in I_1'} t_i \frac{|a_i|}{\theta_p} &\leq \sum_{i \in I_1'} t(G_i) \|G_i x_1\|_{c_0} \\ &\leq \sum_{(0,n_1,\ldots,n_s) \in K_{\delta,p,\eta}} \sum_{h(G)=(0,n_1,\ldots,n_s)} t(G)\|Gx_1\|_{c_0} \\ &\leq \sum_{(0,n_1,\ldots,n_s) \in K_{\delta,p,\eta}} \|x_1\|_{[\mathcal{F}_{n_1},\ldots,\mathcal{F}_{n_s}]} \\ &\leq |K_{\delta,p,\eta}| \|x_1\|_{\mathcal{S}_\eta} < \delta. \end{aligned}$$

The next to last inequality holds since for any $(0, n_1, \ldots, n_s)$, the set $\{G \in \mathcal{T} : h(G) = (0, n_1, \ldots, n_s)\}$ is $[\mathcal{F}_{n_1}, \ldots, \mathcal{F}_{n_s}]$-admissible. Also,

$$(4) \qquad \sum_{i \in I_1'''} t_i \frac{|a_i|}{\theta_p} \leq \sum_{i \in I_1'''} t(G_i) \frac{|a_i|}{\theta_p} \leq \frac{\theta_q}{\varepsilon} \|x_1\|_{\ell^1} = \frac{\theta_q}{\varepsilon\theta_p}.$$

Define $J_1 = I_1'' \cup I_3$, $J_2 = I_1' \cup I_1''' \cup I_2$ and let $\mathcal{T}'$ be the subtree of $\mathcal{T}$ consisting of all nodes in $\bigcup_{i \in I_1''} \mathcal{E}_i$ together with their ancestors. Clearly J_1 is disjoint from J_2. Note that if $E \in \mathcal{L}(\mathcal{T}')$, then $E \in \mathcal{E}_i$ for some $i \in I_1''$. Since $m_i < E < m_{i+1}$ and m_i, m_{i+1} are both contained in the integer interval F_i, $E \subsetneqq F_i$. Hence there exists an immediate successor H of F_i such that $E \subseteq H$. But $h(H) = h(G_i)$ as H and G_i are both immediate successors of F_i. Thus $h(H) \in \mathcal{N}_q \setminus \mathcal{N}_p$. This shows that $\mathcal{T}'$ is (p,q)-restricted. Applying (3) and (4) to (2), we see that

$$\begin{aligned} \mathcal{T}x &\leq \delta + \frac{\theta_q}{\varepsilon\theta_p} + \sum_{i \in I_1'' \cup I_3} t_i \frac{|a_i|}{\theta_p} + \sum_{i \in I_1 \cup I_2} \sum_{E \in \mathcal{E}_i} |a_i| t(E) \|Ez_i\|_{c_0} \\ &= \delta + \frac{\theta_q}{\varepsilon\theta_p} + \sum_{i \in J_1} t_i \frac{|a_i|}{\theta_p} + \Big(\sum_{i \in J_2} + \sum_{i \in I_1''}\Big)\Big(\sum_{E \in \mathcal{E}_i} |a_i| t(E) \|Ez_i\|_{c_0}\Big) \\ &\leq \delta + \frac{\theta_q}{\varepsilon\theta_p} + \mathcal{T}\Big(\sum_{i \in J_1} \frac{a_i}{\theta_p} e_{m_i} + \sum_{i \in J_2} a_i z_i\Big) + \mathcal{T}'\Big(\sum_{i=1}^{r} a_i z_i\Big), \end{aligned}$$

as required. ∎

Assume that X satisfies (†). The next step is to iterate the construction in Proposition 5 to generate vectors with an arbitrary number of layers. The key observation is that these vectors are uniformly bounded. The corresponding layers in the vectors will interact to give the desired finite dimensional ℓ^1 behavior. Let ε be the constant given by condition (†). Suppose (β_n) is the sequence of ordinals increasing to ω^ξ that defines $\mathcal{S}_{\omega^\xi}$. Given any $M_0 \in [\mathbb{N}]$, we choose sequences (p_n), (q_n) in $\mathbb{N}$, a decreasing sequence of infinite subsets (M_n) of M_0 and a sequence of countable ordinals (η_n) less than ω^ξ in the following manner. Pick $p_1 \in \mathbb{N}$ so that $\theta_{p_1} \le \varepsilon^2/4$ and $\gamma(\varepsilon, p_1) + 2 + \beta_1 < \ell(\alpha_{p_1})$. Define $\eta_1 = \gamma(\varepsilon,p_1)+1$. Then choose $q_1 \in \mathbb{N}$ so that $\theta_{q_1} \le \varepsilon\theta_{p_1}/4$. Since $\eta_1+1+\beta_1 < \ell(\alpha_{p_1})$ and $\ell(\alpha_{n_s}\cdots\alpha_{n_1}) < \eta_1$ for all $(0,n_1,\ldots,n_s) \in K_{4^{-1},p_1,\eta_1}$, by the remark following Theorem 1, there exists $M_1 \in [M_0]$ such that $\mathcal{S}_{\beta_1}[\mathcal{S}_{\eta_1+1}] \cap [M_1]^{<\infty} \subseteq \mathcal{F}_{p_1}$ and $[\mathcal{F}_{n_1},\ldots,\mathcal{F}_{n_s}] \cap [M_1]^{<\infty} \subseteq \mathcal{S}_{\eta_1}$ whenever $(0,n_1,\ldots,n_s) \in K_{4^{-1},p_1,\eta_1}$. Assume that the sequences have been chosen up to $n-1$. Pick $p_n > q_{n-1}$ so that $\theta_{p_n} \le \varepsilon^2/4^n$ and

$$\gamma(\varepsilon,p_n) + 2 + \gamma(\varepsilon,q_{n-1}) + 2 + \eta_{n-1} + 1 + \cdots + \eta_1 + 1 + \beta_n < \ell(\alpha_{p_n}).$$

Define $\eta_n = \gamma(\varepsilon,p_n) + \gamma(\varepsilon,q_{n-1}) + 1$. Then choose $q_n > p_n$ so that $\theta_{q_n} \le \varepsilon\theta_{p_n}/4^n$. Since $\eta_n + 1 + \cdots + \eta_1 + 1 + \beta_n < \ell(\alpha_{p_n})$ and $\ell(\alpha_{n_s}\cdots\alpha_{n_1}) < \eta_n$ if $(0,n_1,\ldots,n_s) \in K_{4^{-n},p_n,\eta_n}$, there exists $M_n \in [M_{n-1}]$ so that

$$\mathcal{S}_{\beta_n}[\mathcal{S}_{\eta_1+1},\ldots,\mathcal{S}_{\eta_n+1}] \cap [M_n]^{<\infty} \subseteq \mathcal{F}_{p_n}$$

and $[\mathcal{F}_{n_1},\ldots,\mathcal{F}_{n_s}] \cap [M_n]^{<\infty} \subseteq \mathcal{S}_{\eta_n}$ if $(0,n_1,\ldots,n_s) \in K_{4^{-n},p_n,\eta_n}$. This completes the inductive construction. For every n, let $Z(p_n)$ be the set of all vectors x in c_{00} such that $\|x\|_{\ell^1} = \theta_{p_n}^{-1}$, $\operatorname{supp} x \in \mathcal{S}_{\eta_n+1} \cap [M_n]^{<\infty}$ and $\|x\|_{\mathcal{S}_{\eta_n}} \le 4^{-n}(|K_{4^{-n},p_n,\eta_n}| + 1)^{-1}$. The set $Z(p_n)$ is nonempty by Proposition 3.6 in [10]. Inductively, for $n,k \in \mathbb{N}$, let $Z(p_n,p_{n+1},\ldots,p_{n+k})$ consist of all vectors of the form $\theta_{p_n}^{-1}\sum_{i=1}^r a_i e_{m_i} + \sum_{i=1}^r a_i z_i$, where $m_1 < z_1 < \cdots < m_r < z_r$, $\theta_{p_n}^{-1}\sum_{i=1}^r a_i e_{m_i} \in Z(p_n)$ and $z_i \in Z(p_{n+1},\ldots,p_{n+k})$, $1 \le i \le r$. Recall that an admissible tree $\mathcal{T}$ is said to be (p,q)-restricted if every leaf $E \in \mathcal{T}$ is contained in some node $G \in \mathcal{T}$ with $h(G) \in \mathcal{N}_q \setminus \mathcal{N}_p$. In the following, a (p_0,q_0)-restricted tree is one without any restriction placed on it.

Lemma 6: *Let x be a vector finitely supported in M_n and suppose that $\|x\|_{\mathcal{S}_{\eta_n}} \le 4^{-n}(|K_{4^{-n},p_n,\eta_n}| + 1)^{-1}$. If $0 \le m < n$ and $\mathcal{T}$ is a (p_m,q_m)-restricted admissible tree, then*

$$\mathcal{T}x \le \begin{cases} 4^{-n} + \frac{\theta_{p_n}}{\varepsilon}\|x\|_{\ell^1} & \text{if } m = 0, \\ 4^{-n} + \theta_{p_n}\|x\|_{\ell^1}(4^{-n} + 4^{-m}) & \text{if } 0 < m < n. \end{cases}$$

Proof: First assume that $m = 0$. Observe that $\mathcal{N}_{p_n} \subseteq K_{4^{-n},p_n,\eta_n}$. Indeed, if $(0, n_1, \ldots, n_s) \in \mathcal{N}_{p_n}$, then $\ell(\alpha_{n_s} \cdots \alpha_{n_1}) \leq \gamma(\varepsilon, p_n) < \eta_n$ and $\theta_{n_1} \cdots \theta_{n_s} > \theta_{p_n}/\varepsilon > 4^{-n}\theta_{p_n}$. Thus $(0, n_1, \ldots, n_s) \in K_{4^{-n},p_n,\eta_n}$. For a fixed $(0, n_1, \ldots, n_s)$, $\{E \in \mathcal{L}(\mathcal{T}) : h(E) \in (0, n_1, \ldots, n_s)\}$ is $[\mathcal{F}_{n_1}, \ldots, \mathcal{F}_{n_s}]$-admissible. Hence if $(0, n_1, \ldots, n_s) \in \mathcal{N}_{p_n} \subseteq K_{4^{-n},p_n,\eta_n}$, then

$$\sum_{\substack{E \in \mathcal{L}(\mathcal{T}) \\ h(E)=(0,n_1,\ldots,n_s)}} t(E)\|Ex\|_{c_0} \leq \|x\|_{[\mathcal{F}_{n_1},\ldots,\mathcal{F}_{n_s}]\cap[M_n]^{<\infty}} \leq \|x\|_{\mathcal{S}_{\eta_n}}.$$

Therefore,

$$\begin{aligned}
\mathcal{T}x &\leq \Bigg(\sum_{\substack{E \in \mathcal{L}(\mathcal{T}) \\ h(E) \in \mathcal{N}_{p_n}}} + \sum_{\substack{E \in \mathcal{L}(\mathcal{T}) \\ h(E) \notin \mathcal{N}_{p_n}}} \Bigg) t(E)\|Ex\|_{c_0} \\
&\leq \sum_{(0,n_1,\ldots,n_s) \in K_{4^{-n},p_n,\eta_n}} \sum_{\substack{E \in \mathcal{L}(\mathcal{T}) \\ h(E)=(0,n_1,\ldots,n_s)}} t(E)\|Ex\|_{c_0} + \sum_{\substack{E \in \mathcal{L}(\mathcal{T}) \\ h(E) \notin \mathcal{N}_{p_n}}} \frac{\theta_{p_n}}{\varepsilon}\|Ex\|_{c_0} \\
&\leq |K_{4^{-n},p_n,\eta_n}| \|x\|_{\mathcal{S}_{\eta_n}} + \frac{\theta_{p_n}}{\varepsilon}\|x\|_{\ell^1} \\
&\leq 4^{-n} + \frac{\theta_{p_n}}{\varepsilon}\|x\|_{\ell^1}.
\end{aligned}$$

Assume that $0 < m < n$. If $E \in \mathcal{L}(\mathcal{T})$, pick $G \in \mathcal{T}$ so that $E \subseteq G$ and $h(G) \in \mathcal{N}_{q_m} \setminus \mathcal{N}_{p_m}$. Write $h(G) = (0, n_1, \ldots, n_s)$ and $h(E) = (0, n_1, \ldots, n_t)$, $t \geq s$. If $(0, n_{s+1}, \ldots, n_t) \in \mathcal{N}_{p_n}$, then $\ell(\alpha_{n_t} \cdots \alpha_{n_{s+1}}) \leq \gamma(\varepsilon, p_n)$. Since $h(G) \in \mathcal{N}_{q_m} \subseteq \mathcal{N}_{q_{n-1}}$, we also have $\ell(\alpha_{n_s} \cdots \alpha_{n_1}) \leq \gamma(\varepsilon, q_{n-1})$. Therefore,

$$\begin{aligned}
\ell(\alpha_{n_t} \cdots \alpha_{n_{s+1}} \alpha_{n_s} \cdots \alpha_{n_1}) &= \ell(\alpha_{n_t} \cdots \alpha_{n_{s+1}}) + \ell(\alpha_{n_s} \cdots \alpha_{n_1}) \\
&\leq \gamma(\varepsilon, p_n) + \gamma(\varepsilon, q_{n-1}) < \eta_n.
\end{aligned}$$

It follows that if $(0, n_{s+1}, \ldots, n_t) \in \mathcal{N}_{p_n}$ and $t(E) > 4^{-n}\theta_{p_n}$, then $h(E) \in K_{4^{-n},p_n,\eta_n}$. Thus,

$$\begin{aligned}
&\sum_{\substack{E \in \mathcal{L}(\mathcal{T}) \\ (0,n_{s+1},\ldots,n_t) \in \mathcal{N}_{p_n}}} t(E)\|Ex\|_{c_0} \\
&\leq \sum_{\substack{E \in \mathcal{L}(\mathcal{T}) \\ h(E) \in K_{4^{-n},p_n,\eta_n}}} t(E)\|Ex\|_{c_0} + \sum_{\substack{E \in \mathcal{L}(\mathcal{T}) \\ t(E) \leq 4^{-n}\theta_{p_n}}} t(E)\|Ex\|_{c_0} \qquad (5) \\
&\leq |K_{4^{-n},p_n,\eta_n}| \|x\|_{\mathcal{S}_{\eta_n}} + 4^{-n}\theta_{p_n}\|x\|_{\ell^1} \\
&\leq 4^{-n} + 4^{-n}\theta_{p_n}\|x\|_{\ell^1}.
\end{aligned}$$

On the other hand, if $(0, n_{s+1}, \ldots, n_t) \notin \mathcal{N}_{p_n}$, then $\varepsilon\theta_{n_{s+1}} \cdots \theta_{n_t} \leq \theta_{p_n}$. Similarly, $\varepsilon\theta_{n_1} \cdots \theta_{n_s} \leq \theta_{p_m}$ since $h(G) \notin \mathcal{N}_{p_m}$. Hence

$$t(E) = \theta_{n_1} \cdots \theta_{n_s}\theta_{n_{s+1}} \cdots \theta_{n_t} \leq \theta_{p_m}\theta_{p_n}/\varepsilon^2.$$

Thus

$$\sum_{\substack{E\in\mathcal{L}(\mathcal{T})\\(0,n_{s+1},\dots,n_t)\notin\mathcal{N}_{p_n}}} t(E)\|Ex\|_{c_0} \le \frac{\theta_{p_m}\theta_{p_n}}{\varepsilon^2}\|x\|_{\ell^1} \le 4^{-m}\theta_{p_n}\|x\|_{\ell^1}. \tag{6}$$

Combining (5) and (6) completes the proof. ∎

LEMMA 7: *Let x be a vector in $Z(p_n,\dots,p_{n+k})$, where $n\in\mathbb{N}$ and $k\in\mathbb{N}\cup\{0\}$. If $0\le m<n$ and $\mathcal{T}$ is a (p_m,q_m)-restricted admissible tree, then*

$$\mathcal{T}x \le 4^{-(n-1)}\sum_{j=0}^{k}2^{-j} + \begin{cases}\frac{1}{\varepsilon}-3\cdot 4^{-(n+k)} & \text{if } m=0,\\ 4^{-m} & \text{if } 0<m<n.\end{cases}$$

Proof: Observe that any vector $x\in Z(p_n)$ satisfies the hypothesis of Lemma 6 and that $\|x\|_{\ell^1}=\theta_{p_n}^{-1}$. The result for $k=0$ follows from the same lemma.

Now suppose the result holds for some k and consider a vector

$$x\in Z(p_n,\dots,p_{n+k+1})$$

and a (p_m,q_m)-restricted admissible tree $\mathcal{T}$, $0\le m<n$. Write

$$x=\theta_{p_n}^{-1}\sum_{i=1}^{r}a_ie_{m_i}+\sum_{i=1}^{r}a_iz_i=x_1+x_2$$

according to the definition of $Z(p_n,\dots,p_{n+k+1})$. One can easily verify all the conditions preceding Proposition 5 with the parameters $\delta=4^{-n}$, $p=p_n$, $q=q_n$, $M=M_n$, and $\eta=\eta_n$. By Proposition 5, we obtain a (p_n,q_n)-restricted admissible tree $\mathcal{T}'$ and disjoint sets J_1 and J_2 so that

$$\begin{aligned}\mathcal{T}x &\le \mathcal{T}\Big(\sum_{i\in J_1}\frac{a_i}{\theta_{p_n}}e_{m_i}+\sum_{i\in J_2}a_iz_i\Big)+\mathcal{T}'x_2+4^{-n}+\frac{\theta_{q_n}}{\varepsilon\theta_{p_n}}\\ &\le \mathcal{T}\Big(\sum_{i\in J_1}\frac{a_i}{\theta_{p_n}}e_{m_i}+\sum_{i\in J_2}a_iz_i\Big)+\mathcal{T}'x_2+2\cdot 4^{-n}.\end{aligned}$$

By Lemma 6,

$$\mathcal{T}\Big(\sum_{i\in J_1}\frac{a_i}{\theta_{p_n}}e_{m_i}\Big)\le\begin{cases}4^{-n}+\frac{1}{\varepsilon}\sum_{i\in J_1}|a_i| & \text{if } m=0,\\ 4^{-n}+(4^{-n}+4^{-m})\sum_{i\in J_1}|a_i| & \text{if } m\ne 0.\end{cases}$$

Moreover, by the inductive hypothesis,

$$\begin{aligned}\mathcal{T}\Big(\sum_{i\in J_2}a_iz_i\Big) &\le \sum_{i\in J_2}|a_i|\sup_{i\in J_2}\mathcal{T}z_i\\ &\le \sum_{i\in J_2}|a_i|\left(4^{-n}\sum_{j=0}^{k}2^{-j}+\begin{cases}\frac{1}{\varepsilon}-3\cdot 4^{-(n+k+1)} & \text{if } m=0,\\ 4^{-m} & \text{if } m\ne 0.\end{cases}\right)\end{aligned}$$

Using the fact that

$$u\sum_{i\in J_1}|a_i|+v\sum_{i\in J_2}|a_i|\le\max\{u,v\}\sum_{i\in J_1\cup J_2}|a_i|\le\max\{u,v\}$$

if $u,v\ge 0$, we see that

$$\begin{aligned}\mathcal{T}\Big(\sum_{i\in J_1}\frac{a_i}{\theta_{p_n}}e_{m_i}+\sum_{i\in J_2}a_iz_i\Big)&\\ \le 4^{-n}+\sum_{i\in J_1}|a_i|\Bigg(&\begin{cases}\frac{1}{\varepsilon} & \text{if } m=0\\ 4^{-n}+4^{-m} & \text{if } m\ne 0\end{cases}\Bigg)\\ &+\sum_{i\in J_2}|a_i|\Bigg(4^{-n}\sum_{j=0}^{k}2^{-j}+\begin{cases}\frac{1}{\varepsilon}-3\cdot 4^{-(n+k+1)} & \text{if } m=0,\\ 4^{-m} & \text{if } m\ne 0\end{cases}\Bigg)\\ \le 4^{-n}+4^{-n}\sum_{j=0}^{k}2^{-j}+&\begin{cases}\frac{1}{\varepsilon}-3\cdot 4^{-(n+k+1)} & \text{if } m=0,\\ 4^{-m} & \text{if } m\ne 0.\end{cases}\end{aligned}$$

Since $\mathcal{T}'$ is (p_n,q_n)-restricted, the inductive hypothesis yields

$$\mathcal{T}'x_2\le 4^{-n}\sum_{j=0}^{k}2^{-j}+4^{-n}.$$

Therefore,

$$\begin{aligned}\mathcal{T}x\le&4^{-n}+4^{-n}\sum_{j=0}^{k}2^{-j}+\begin{cases}\frac{1}{\varepsilon}-3\cdot 4^{-(n+k+1)} & \text{if } m=0\\ 4^{-m} & \text{if } m\ne 0\end{cases}\\ &+4^{-n}\sum_{j=0}^{k}2^{-j}+4^{-n}+2\cdot 4^{-n}\\ =&4\cdot 4^{-n}+2\cdot 4^{-n}\sum_{j=0}^{k}2^{-j}+\begin{cases}\frac{1}{\varepsilon}-3\cdot 4^{-(n+k+1)} & \text{if } m=0\\ 4^{-m} & \text{if } m\ne 0\end{cases}\\ =&4^{-(n-1)}+4^{-(n-1)}\sum_{j=0}^{k}2^{-(j+1)}+\begin{cases}\frac{1}{\varepsilon}-3\cdot 4^{-(n+k+1)} & \text{if } m=0\\ 4^{-m} & \text{if } m\ne 0\end{cases}\\ =&4^{-(n-1)}\sum_{j=0}^{k+1}2^{-j}+\begin{cases}\frac{1}{\varepsilon}-3\cdot 4^{-(n+k+1)} & \text{if } m=0,\\ 4^{-m} & \text{if } m\ne 0.\end{cases}\qquad\blacksquare\end{aligned}$$

The case $m=0$ gives the next corollary.

COROLLARY 8: *The set $Z(p_n,p_{n+1},\dots,p_{n+k})$ has norm bounded by $2\cdot 4^{-(n-1)}+1/\varepsilon$.*

PROPOSITION 9: *Let x be a vector in $Z(p_n, \ldots, p_{n+k})$, where $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$. Then there exists a sequence of pairwise disjoint vectors $(y_j)_{j=0}^k$ such that*

$$x = \sum_{j=0}^{k} y_j, \quad \|y_j\|_{\ell^1} = \frac{1}{\theta_{p_{n+j}}}, \quad 0 \le j \le k$$

and

$$\operatorname{supp} y_j \in [\mathcal{S}_{\eta_n+1}, \ldots, \mathcal{S}_{\eta_{n+j}+1}] \cap [M_{n+j}]^{<\infty}.$$

Proof: The proof is by induction on k. If $k = 0$, set $y_0 = x$ and the claim is clear. Assume the proposition holds for some k and consider a vector $x \in Z(p_n, \ldots, p_{n+k+1})$. Write $x = \theta_{p_n}^{-1} \sum_{i=1}^r a_i e_{m_i} + \sum_{i=1}^r a_i z_i$ according to the definition of $Z(p_n, \ldots, p_{n+k+1})$. By the inductive hypothesis, for each i, there is a sequence of pairwise disjoint vectors $(y_j^i)_{j=1}^{k+1}$ such that $z_i = \sum_{j=1}^{k+1} y_j^i$, $\|y_j^i\|_{\ell^1} = \theta_{p_{n+j}}^{-1}$, $\operatorname{supp} y_j^i \in [\mathcal{S}_{\eta_{n+1}+1}, \ldots, \mathcal{S}_{\eta_{n+j}+1}] \cap [M_{n+j}]^{<\infty}$, $1 \le j \le k+1$. Set $y_0 = \theta_{p_n}^{-1} \sum_{i=1}^r a_i e_{m_i}$, and $y_j = \sum_{i=1}^r a_i y_j^i$, $1 \le j \le k+1$. Then $(y_j)_{j=0}^{k+1}$ is a pairwise disjoint sequence such that $\sum_{j=0}^{k+1} y_k = x$. Clearly, $\|y_j\|_{\ell^1} = \sum_{i=1}^r |a_i| \|y_j^i\|_{\ell^1} = \theta_{p_{n+j}}^{-1}$, $1 \le j \le k+1$, and $\|y_0\|_{\ell^1} = \theta_{p_n}^{-1} \sum_{i=1}^r |a_i| = \theta_{p_n}^{-1}$. Also, $\operatorname{supp} y_0 \in \mathcal{S}_{\eta_n+1} \cap [M_n]^{<\infty}$ since $y_0 \in Z(p_n)$. Furthermore, since $m_1 < y_j^1 < \cdots < m_r < y_j^r$ and $\{m_1, \ldots, m_r\} \in \mathcal{S}_{\eta_n+1}$,

$$\operatorname{supp} y_j \in [\mathcal{S}_{\eta_n+1}, \ldots, \mathcal{S}_{\eta_{n+j}+1}] \cap [M_{n+j}]^{<\infty}, \quad 1 \le j \le k+1. \qquad \blacksquare$$

Proof of Theorem 4: Beginning with $M_0 = M$, carry out the construction above. Now take a block basis (z_k) of $(e_n)_{n \in M}$ such that $z_k \in Z(p_1, p_2, \ldots, p_k)$ for all k. By Corollary 8, $\|z_k\| \le 2 + 1/\varepsilon$ for all k. Suppose $F \in \mathcal{S}_{\omega^\xi}$. Then there exists $j_0 \le \min F$ such that $F \in \mathcal{S}_{\beta_{j_0}}$. By Proposition 9, for all $k \in F$, there exists y_k such that $|y_k| \le |z_k|$, $\|y_k\|_{\ell^1} = \theta_{p_{j_0}}^{-1}$ and

$$\operatorname{supp} y_k \in [\mathcal{S}_{\eta_1+1}, \ldots, \mathcal{S}_{\eta_{j_0}+1}] \cap [M_{j_0}]^{<\infty}.$$

Thus for all scalars (a_k),

$$\Big\| \sum_{k \in F} a_k z_k \Big\| \ge \Big\| \sum_{k \in F} a_k y_k \Big\| \ge \theta_{p_{j_0}} \Big\| \sum_{k \in F} a_k y_k \Big\|_{\mathcal{F}_{p_{j_0}}} = \theta_{p_{j_0}} \Big\| \sum_{k \in F} a_k y_k \Big\|_{\ell^1},$$

as $\mathcal{S}_{\beta_{j_0}}[\mathcal{S}_{\eta_1+1}, \ldots, \mathcal{S}_{\eta_{j_0}+1}] \cap [M_{j_0}]^{<\infty} \subseteq \mathcal{F}_{p_{j_0}}$. Therefore,

$$\Big\| \sum_{k \in F} a_k z_k \Big\| \ge \theta_{p_{j_0}} \sum_{k \in F} |a_k| \|y_k\|_{\ell^1} = \sum_{k \in F} |a_k|. \qquad \blacksquare$$

In the rest of the section, we prove the converse to Theorem 4. By [9, Proposition 1], we may assume without loss of generality that there exists a sequence $(\ell_n) \subseteq \mathbb{N}$ converging to ∞ such that $\mathcal{F}_n = (\mathcal{F}_n \cap [\mathbb{N}_{\ell_n}]^{<\infty}) \cup \mathcal{S}_0$ for all $n \in \mathbb{N}$, where $\mathbb{N}_k = \{n \in \mathbb{N} : n \geq k\}$.

LEMMA 10: *If (†) fails, then for all $\varepsilon > 0$ and all $M \in [\mathbb{N}]$, there exist $M' \in [M]$ and a regular family $\mathcal{H}$ containing $\mathcal{S}_0$, $\iota(\mathcal{H}) < \omega^{\omega^\xi}$, such that for all sufficiently large m, there exist $n_1, \ldots, n_s$ so that $\varepsilon\theta_{n_1}\cdots\theta_{n_s} > \theta_m$ and $\mathcal{F}_m \cap [M']^{<\infty} \subseteq [\mathcal{H}, \mathcal{F}_{n_1}, \ldots, \mathcal{F}_{n_s}]$.*

Proof: Fix $\varepsilon > 0$. Since (†) fails, there exists $\beta < \omega^\xi$ such that for all m, $\gamma(\varepsilon, m) + 2 + \beta \geq \ell(\alpha_m)$. Therefore, for all large enough m, say $m > m_0$, there exist $n_1, \ldots, n_s$ such that $\varepsilon\theta_{n_1}\cdots\theta_{n_s} > \theta_m$ and $\ell(\alpha_{n_s}\cdots\alpha_{n_1}) + 2 + \beta \geq \ell(\alpha_m)$. Let $\beta' = 2 + \beta + 1 < \omega^\xi$. Then $\ell(\alpha_m) < \ell(\alpha_{n_s}\cdots\alpha_{n_1}) + \beta'$. Thus,

$$\iota(\mathcal{F}_m) < \iota(\mathcal{S}_{\beta'}[\mathcal{F}_{n_1}, \ldots, \mathcal{F}_{n_s}]).$$

By the remark after Theorem 1, for all $N \in [\mathbb{N}]$, there exists $N' \in [N]$ such that

$$\mathcal{F}_m \cap [N']^{<\infty} \subseteq \mathcal{S}_{\beta'}[\mathcal{F}_{n_1}, \ldots, \mathcal{F}_{n_s}].$$

Given $M \in [\mathbb{N}]$, applying the above argument repeatedly, we obtain infinite sets

$$M \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_k \supseteq \cdots$$

such that for all $k \in \mathbb{N}$, there exist $n_1, \ldots, n_s$ (depending on k) such that $\varepsilon\theta_{n_1}\cdots\theta_{n_s} > \theta_{m_0+k}$ and $\mathcal{F}_{m_0+k} \cap [M_k]^{<\infty} \subseteq \mathcal{S}_{\beta'}[\mathcal{F}_{n_1}, \ldots, \mathcal{F}_{n_s}]$. Choose (m_k) so that $m_0 < m_1 < m_2 < \cdots$. and $m_k \in M_k$ for all $k \in \mathbb{N}$. Let $M' = (m_k)_{k=1}^\infty$. For all $k \in \mathbb{N}$, define $\mathcal{B}_k = \{G : \ell_{m_0+k} \leq G, |G| \leq m_k\}$ and $\mathcal{B} = \bigcup_{k=1}^\infty \mathcal{B}_k \cup \mathcal{S}_0$. Let $\mathcal{H} = (\mathcal{B}, \mathcal{S}_{\beta'})$. Then $\mathcal{H}$ contains $\mathcal{S}_0$ and

$$\iota(\mathcal{H}) = \iota(\mathcal{B}, \mathcal{S}_{\beta'}) = \iota(\mathcal{S}_{\beta'}) + \iota(\mathcal{B}) = \omega^{\beta'} + \omega < \omega^{\omega^\xi}.$$

Consider a set $F \in \mathcal{F}_{m_0+k} \cap [M']^{<\infty}$ for some $k \in \mathbb{N}$. Write $F = F_1 \cup F_2$, where $F_1 = F \cap [1, m_k)$ and $F_2 = F \cap [m_k, \infty)$. Since

$$F_1 \in \mathcal{F}_{m_0+k} = (\mathcal{F}_{m_0+k} \cap [\mathbb{N}_{\ell_{m_0+k}}]^{<\infty}) \cup \mathcal{S}_0,$$

either $F_1 \in \mathcal{S}_0 \subseteq \mathcal{B}$ or $F_1 \in \mathcal{F}_{m_0+k} \cap [\mathbb{N}_{\ell_{m_0+k}}]^{<\infty}$. In the latter case, $\ell_{m_0+k} \leq F_1$ and $|F_1| \leq m_k$ and hence $F_1 \in \mathcal{B}_k \subseteq \mathcal{B}$. Also, $F_2 \in \mathcal{F}_{m_0+k} \cap [M_k]^{<\infty}$ implies that there exist $n_1, \ldots, n_s$ such that $\varepsilon\theta_{n_1}\cdots\theta_{n_s} > \theta_{m_0+k}$ and $F_2 \in \mathcal{S}_{\beta'}[\mathcal{F}_{n_1}, \ldots, \mathcal{F}_{n_s}]$. Therefore, $F \in (\mathcal{B}, \mathcal{S}_{\beta'})[\mathcal{F}_{n_1}, \ldots, \mathcal{F}_{n_s}] = \mathcal{H}[\mathcal{F}_{n_1}, \ldots, \mathcal{F}_{n_s}]$. ∎

PROPOSITION 11 ([9, Proposition 14]): *Suppose for all $\varepsilon > 0$, there exist a regular family $\mathcal{G}_\varepsilon$ and $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, there exist $n_1, \ldots, n_s \in \mathbb{N}$ satisfying $\theta_m < \varepsilon\theta_{n_1} \cdots \theta_{n_s}$ and $\mathcal{F}_m \subseteq [\mathcal{G}_\varepsilon, \mathcal{F}_{n_1}, \ldots, \mathcal{F}_{n_s}]$. Then*

$$I_b(X) \leq \sup_{\varepsilon>0} \sup_{n\in\mathbb{N}} [\iota(\mathcal{G}_\varepsilon) \cdot \alpha_n^\omega].$$

THEOREM 12: *Suppose that (†) fails; then for all $M \in [\mathbb{N}]$, there exists $N \in [M]$ such that*

$$I([(e_k)_{k\in N}]) = \omega^{\omega^\xi}.$$

In particular, $[(e_k)_{k\in N}]$ does not contain any ℓ^1-$\mathcal{S}_{\omega^\xi}$-spreading model.

Proof: By Lemma 10, there exist infinite sets $M \supseteq M_1 \supseteq \cdots \supseteq M_k \supseteq \cdots$ such that for all $i \in \mathbb{N}$, there exists a regular family $\mathcal{H}_i$ containing $\mathcal{S}_0$, $\iota(\mathcal{H}_i) < \omega^{\omega^\xi}$, such that for all sufficiently large n, say $n \geq m_0(i)$, there exist $n_1, \ldots, n_s$ so that $\theta_n < \theta_{n_1} \cdots \theta_{n_s}/i$ and $\mathcal{F}_n \cap [M_i]^{<\infty} \subseteq [\mathcal{H}_i, \mathcal{F}_{n_1}, \ldots, \mathcal{F}_{n_s}]$. Choose $m_1 < m_2 < m_3 < \cdots$ such that $m_k \in M_k$ and let $N = (m_k)$. Set $Y = [(e_k)_{k\in N}]$. Note that $Y = T[(\theta_n, \mathcal{G}_n)_{n=1}^\infty]$, where $G \in \mathcal{G}_n$ if and only if $\{m_k : k \in G\} \in \mathcal{F}_n$.

Suppose $\varepsilon > 0$ is given. Pick $i \in \mathbb{N}$ such that $1/i < \varepsilon$. Assume that $n \geq m_0(i)$ and $\ell_n \geq m_i$. If $G \in \mathcal{G}_n$, then $F = \{m_k : k \in G\} \in \mathcal{F}_n \cap [N]^{<\infty}$. Since $\mathcal{F}_n = (\mathcal{F}_n \cap [\mathbb{N}_{\ell_n}]^{<\infty}) \cup \mathcal{S}_0$, either $F \in \mathcal{S}_0$ or $F \in \mathcal{F}_n \cap [\mathbb{N}_{\ell_n}]^{<\infty}$. In the latter case, $F \geq \ell_n \geq m_i$ and thus $F \in \mathcal{F}_n \cap [M_i]^{<\infty}$. Hence in either case, $F \in [\mathcal{H}_i, \mathcal{F}_{n_1}, \ldots, \mathcal{F}_{n_s}]$ for some $n_1, \ldots, n_s$ such that $\theta_n < \varepsilon\theta_{n_1} \cdots \theta_{n_s}$. Therefore,

$$\mathcal{G}_n \subseteq [\mathcal{J}_i, \mathcal{G}_{n_1}, \ldots, \mathcal{G}_{n_s}],$$

where $G \in \mathcal{J}_i$ if and only if $\{m_k : k \in G\} \in \mathcal{H}_i$. Note that $\iota(\mathcal{J}_i) < \omega^{\omega^\xi}$. Thus, according to Proposition 11,

$$I_b(Y) \leq \sup_i \sup_{n\in\mathbb{N}} [\iota(\mathcal{J}_i) \cdot \alpha_n^\omega] = \sup_i [\iota(\mathcal{J}_i) \cdot \omega^{\omega^\xi}] = \omega^{\omega^\xi}.$$

However, $I_b(Y) \geq \omega^{\omega^\xi}$ by part 1 of [9, Theorem 14]. Hence $I_b(Y) = \omega^{\omega^\xi}$. Finally, $I_b(Y) = I(Y)$ by [6, Corollary 5.13] since $I_b(Y) \geq \omega^\omega$. By [6, Lemma 5.11], $I(Y, K) < \omega^{\omega^\xi}$. Thus Y does not contain any ℓ^1-$\mathcal{S}_{\omega^\xi}$-spreading model. ∎

3. Mixed Tsirelson spaces constructed with Schreier families

In this section, we apply the results of the last section to mixed Tsirelson spaces of the type $T[(\theta_n, \mathcal{S}_{\beta_n})_{n=1}^\infty]$, where (θ_n) is a nonincreasing null sequence in $(0,1)$, $\sup_n \beta_n = \omega^\xi > \beta_n > 0$ for all $n \in \mathbb{N}$, and $0 < \xi < \omega_1$. In the present situation, the function γ is given by

$$\gamma(\varepsilon, m) = \max\{\beta_{n_s} + \cdots + \beta_{n_1} : \varepsilon\theta_{n_s} \cdots \theta_{n_1} > \theta_m\} \quad (\max \emptyset = 0).$$

Theorems 4 and 12 give

THEOREM 13: *Let (β_n) be as above and let (e_n) be the unit vector basis of the mixed Tsirelson space $T[(\theta_n, \mathcal{S}_{\beta_n})_{n=1}^\infty]$. If condition (†) holds, then for any $M \in [\mathbb{N}]$, $(e_n)_{n\in M}$ contains an ℓ^1-$\mathcal{S}_{\omega^\xi}$-spreading model. If condition (†) fails, then for all $M \in [\mathbb{N}]$, there exists $N \in [M]$ such that $[(e_k)_{k\in N}]$ does not contain any ℓ^1-$\mathcal{S}_{\omega^\xi}$-spreading model.*

In the event that the Schreier families $\mathcal{S}_\beta$, β a limit ordinal, are defined using special choices, the second part of Theorem 13 can be strengthened. The special "standard" choices are described as follows. For all limit ordinals $\alpha < \omega_1$, fix a sequence of ordinals strictly increasing to α. If $\beta = \omega^{\beta_1} \cdot m_1 + \cdots + \omega^{\beta_k} \cdot m_k$ is a limit ordinal, determine $\mathcal{S}_\beta$ using the sequence

$$\hat{\beta}_n = \begin{cases} \omega^{\beta_1} \cdot m_1 + \cdots + \omega^{\beta_k} \cdot (m_k - 1) + \omega^{\beta_k - 1} \cdot n & \text{if } \beta_k \text{ is a successor,} \\ \omega^{\beta_1} \cdot m_1 + \cdots + \omega^{\beta_k} \cdot (m_k - 1) + \omega^{\zeta_n} & \text{if } \beta_k \text{ is a limit,} \end{cases}$$

where (ζ_n) is the chosen sequence of ordinals increasing to β_k.

THEOREM 14 ([9, Theorem 26]): *Follow the notation above and apply the standard choices to define Schreier families. If there exists $\varepsilon > 0$ such that for all $\beta < \omega^\xi$, there exists $m \in \mathbb{N}$ satisfying $\gamma(\varepsilon, m) + 2 + \beta < \beta_m$, then $I_b(T[\theta_n, \mathcal{S}_{\beta_n})_{n=1}^\infty]) = \omega^{\omega^\xi \cdot 2}$. Otherwise, $I_b(T(\mathcal{F}_0, (\theta_n, \mathcal{S}_{\beta_n})_{n=1}^\infty)) = \omega^{\omega^\xi}$.*

For "standard" Schreier families, the second part of Theorem 13 can be improved.

THEOREM 15: *Let (β_n) be as above and apply the standard choices to define Schreier families. If (†) fails, then $I(T[(\theta_n, \mathcal{S}_{\beta_n})_{n=1}^\infty]) = \omega^{\omega^\xi}$. In particular, $T[(\theta_n, \mathcal{S}_{\beta_n})_{n=1}^\infty]$ does not contain any ℓ^1-$\mathcal{S}_{\omega^\xi}$-spreading model.*

Note that for finite β_n's, no choices need to be made in defining the Schreier families $\mathcal{S}_n$. It is worthwhile to record the result in this case.

THEOREM 16: *If* $\theta_{m+n} \geq \theta_m\theta_n$ *for all* m, n *and* $\lim_m \limsup_n \theta_{m+n}/\theta_n > 0$, *then* $[(e_{k_n})]$ *contains an* ℓ^1-$\mathcal{S}_\omega$*-spreading model for any subsequence* (e_{k_n}) *of the unit vector basis* (e_k) *of* $T[(\theta_n, \mathcal{S}_n)_{n=1}^\infty]$. *Otherwise* $T[(\theta_n, \mathcal{S}_n)_{n=1}^\infty]$ *contains no* ℓ^1-$\mathcal{S}_\omega$*-spreading model.*

Remark: It can be shown that for sequences (θ_n) such that $\theta_{m+n} \geq \theta_m\theta_n$ for all m, n, the condition $\lim_m \limsup_n \theta_{m+n}/\theta_n > 0$ is strictly weaker than the condition $\lim \theta_n^{1/n} = 1$.

References

[1] D. E. Alspach and S. Argyros, *Complexity of weakly null sequences*, Dissertationes Mathematicae **321** (1992), 1–44.

[2] S. A. Argyros and I. Deliyanni, *Examples of asymptotic* ℓ^1 *Banach spaces*, Transactions of the American Mathematical Society **349** (1997), 973–995.

[3] S. A. Argyros, I. Deliyanni, D. N. Kutzarova and A. Manoussakis, *Modified mixed Tsirelson spaces*, Journal of Functional Analysis **159** (1998), 43–109.

[4] S. A. Argyros, I. Deliyanni and A. Manoussakis, *Distortion and spreading models in modified mixed Tsirelson spaces*, Studia Mathematica **157** (2003), 199–236.

[5] I. Gasparis, *A dichotomy theorem for subsets of the power set of the natural numbers*, Proceedings of the American Mathematical Society **129** (2001), 759–764.

[6] R. Judd and E. Odell, *Concerning the Bourgain* ℓ_1 *index of a Banach space*, Israel Journal of Mathematics **108** (1998), 145–171.

[7] D. Kutzarova and P.-K. Lin, *Remarks about Schlumprecht space*, Proceedings of the American Mathematical Society **128** (2000), 2059–2068.

[8] D. Leung and W.-K. Tang, *The* ℓ^1*-indices of Tsirelson type spaces*, Proceedings of the American Mathematical Society **131** (2003), 511–521.

[9] D. Leung and W.-K. Tang, *The Bourgain* ℓ^1*-index of mixed Tsirelson space*, Journal of Functional Analysis **199** (2003), 301–331.

[10] E. Odell, N. Tomczak-Jaegermann and R. Wagner, *Proximity to* ℓ_1 *and distortion in asymptotic* ℓ_1 *spaces*, Journal of Functional Analysis **150** (1997), 101–145.

[11] T. Schlumprecht, *An arbitrarily distortable Banach space*, Israel Journal of Mathematics **76** (1991), 81–95.